MATHEMATICS SPECIALIST

MAWA Year 12 Examination 2020

Calculator-free

Marking Key

The release date for this exam and marking scheme is 12th June.

Question1 **(4 marks)**

Solution	
If $$\frac{z}{\sqrt{3}-i}=\frac{1}{\sqrt{3}+i} \Rightarrow z=\frac{(\sqrt{3}-i)(\sqrt{3}-i)}{(\sqrt{3}+i)(\sqrt{3}-i)}=\frac{2-2\sqrt{3}i}{3+1}=\frac{1}{2}(1-\sqrt{3}i)$$ Hence $$z=\frac{1}{2}\left(1-\sqrt{3}i\right)$$ We note that $z=rcis(\theta)$ where $r=1$ and $\theta=-\pi/3$	
Mathematical behaviours	**Marks**
• multiplies fraction by complex conjugate of the denominator	1
• evaluates z correctly	1
• converts to polar form for modulus and argument	1+1

Question 2 (9 marks)

Question 2(a) **(4 marks)**

Solution	
The system of equations first reduces to $$x+1.5y+z=5$$ $$3x+4y-z=4 \quad (*)$$ $$x+y+az=b$$ and then to $$x+1.5y+z=5$$ $$-0.5y-4z=-11 \quad (**)$$ $$-0.5y+(a-1)z=b-5$$ and in turn $$x+1.5y+z=5$$ $$y+8z=22$$ $$(a+3)z=b+6$$ Hence there are infinitely many solutions when $a=-3$ and $b=-6$	
Mathematical behaviours	**Marks**
• eliminates one variable (*)	1
• eliminates a second variable (**)	1
• obtains correct values for a and b	1+1

Question 2(b) **(2 marks)**

Solution	
There is no solution if $a-3$ and $b \neq -6$	
Mathematical behaviours	Marks
• states correct value for a • states correct values for b	1 1

Question 2(c) **(3 marks)**

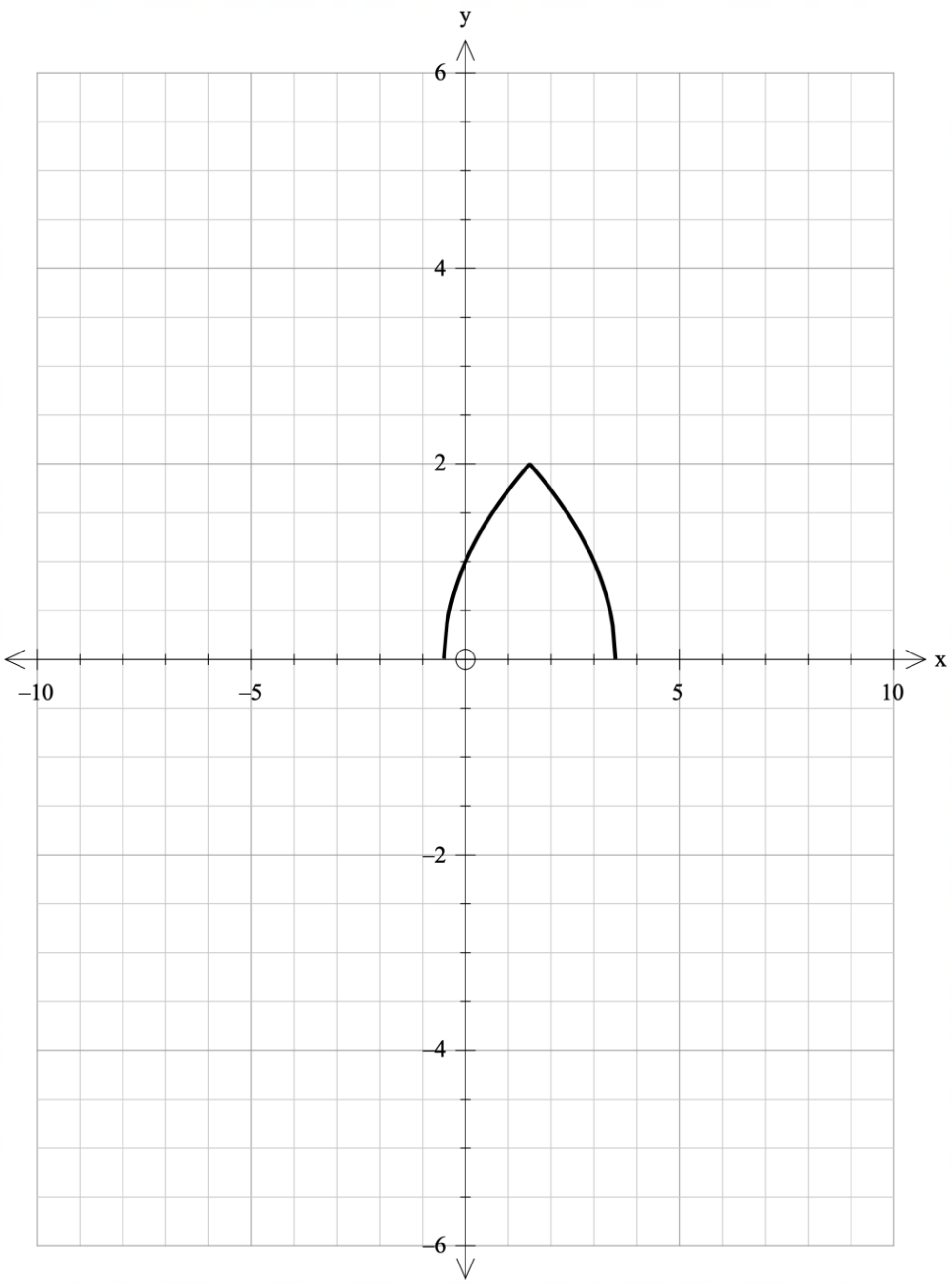
Solution	
If $a=2$ and $b=-5$ then $5z=1$ and hence $z=1/5$ Now $y+8z=22$ so $y=102/5$ Lastly $x+y+2z=-5$ so $x=(-102-2-25)/5=-129/5$ Hence $$(x,y,z)=\left(-\frac{129}{5},\frac{102}{5},\frac{1}{5}\right).$$	
Mathematical behaviours	Marks
• obtains correct value for z • obtains correct value for y • obtains correct value for x	1 1 1

Question 3 **(6 marks)**

Question 3(a) **(3 marks)**

Solution	
The function $f(x)=\sqrt{4-\|3-2x\|}$ is defined if $4-\|3-2x\|\geq 0$ If $3-2x=4 \Rightarrow x=-0.5$ If $3-2x=-4 \Rightarrow x=3.5$ This means that the function is defined for $x\in[-0.5,3.5]$	
Mathematical behaviours	Marks
• derives the positivity requirement for $4-\|3-2x\|$ • evaluates the lower and upper limits of the domain	1 1+1

Question 3(b) **(3 marks)**

Solution


Mathematical behaviours	Marks
• indicates the general shape of the graph • shows the maximum value at (1.5,2) • makes it clear that the function is not differentiable at the max point	1 1 1

Question 4 (5 marks)

Question 4(a) (2 marks)

Solution	
The sphere is given by $$x^2 - 4x + 4 + y^2 + z^2 + 10z + 25 = 20 + 4 + 25$$ i.e. $$(x-2)^2 + y^2 + (z+5)^2 = 7^2$$ which is a sphere centre $(2, 0, -5)$ and of radius 7	
Mathematical behaviours	**Marks**
• completes the square • states the correct centre and radius	1 1

Question 4(b) (3 marks)

Solution	
The centre of the sphere C has co-ordinates (2,0,-5) so that $$\overrightarrow{CA} = (-2, 3, 6)$$ This is normal to the tangent plane so the equation of this plane is of the form $-2x + 3y + 6z = d$ As A (0,3,1) is on the plane, $$d = 9 + 6 = 15$$ Hence required plane is $$-2x + 3y + 6z = 15$$ Alternatively this solution could be written in normal form so $$r.(-2, 3, 6) = 15$$	
Mathematical behaviours	**Marks**
• calculates the normal $\overrightarrow{CA}$ correctly • writes down general equation of the plane with the unknown constant • obtains a correct equation for the tangent plane	1 1 1

Question 5 (5 marks)

Question 5(a) (2 marks)

Solution
Let $z=\alpha+i\beta$ in which case $\bar{z}=\alpha-i\beta$; then $z\bar{z}=(\alpha+i\beta)(\alpha-i\beta)=\alpha^2+\beta^2=\|z\|^2$

Mathematical behaviours	Marks
• writes down an appropriate form for z and hence $\bar{z}$	1
• multiplies terms to prove the required result	1

Question 5(b) (3 marks)

Solution
If a and b are complex numbers then part (a) shows that $$\|a+b\|^2=(a+b)(\bar{a}+\bar{b})=\|a\|^2+\|b\|^2+a\bar{b}+\bar{a}b$$ Similarly $$\|a-b\|^2=(a-b)(\bar{a}-\bar{b})=\|a\|^2+\|b\|^2-a\bar{b}-\bar{a}b$$ Adding the results implies that $$\|a+b\|^2+\|a-b\|^2=2\left\{\|a\|^2+\|b\|^2\right\}.$$

Mathematical behaviours	Marks
• derives expression for $\|a+b\|^2$	1
• derives expression for $\|a-b\|^2$	1
• draws the required conclusion	1

Question 6 **(6 marks)**

Question 6(a) **(2 marks)**

Solution
The vector equation of the line containing BE is $\boldsymbol{r} = \overrightarrow{OE} + t\overrightarrow{EB}$ Now $\overrightarrow{OE} = \frac{1}{2}\overrightarrow{OA} = \frac{1}{2}\boldsymbol{a}$ and $\overrightarrow{EB} = \overrightarrow{OB} - \overrightarrow{OE} = \boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$ so the required equation is $$\boldsymbol{r} = \frac{1}{2}\boldsymbol{a} + t\left(\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}\right) = \frac{1-t}{2}\boldsymbol{a} + t\boldsymbol{b}$$

Mathematical behaviours	Marks
• obtains $\overrightarrow{EB}$ correctly • derives correct equation	1 1

Question 6(b) **(4 marks)**

Solution
$\overrightarrow{OF} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a}) = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$ so the vector equation of the line containing OF is $\boldsymbol{r}' = \frac{t'}{2}(\boldsymbol{a} + \boldsymbol{b})$ Solving $\boldsymbol{r} = \boldsymbol{r}'$ gives $\frac{1}{2}\boldsymbol{a} + t\left(\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}\right) = \frac{1-t}{2}\boldsymbol{a} + t\boldsymbol{b} = \frac{t'}{2}(\boldsymbol{a} + \boldsymbol{b})$ (*) and so $\frac{1-t}{2} = \frac{t'}{2}$ and $t = \frac{t'}{2}$ and so $t = \frac{1}{3}$ and $t' = \frac{2}{3}$ So $\overrightarrow{OC} = \frac{1-1/3}{2}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$

Mathematical behaviours	Marks
• obtains equation of a second median line • obtains equation (*) (or equivalent) • determines value of t and t' • completes proof in a valid way	1 1 1 1

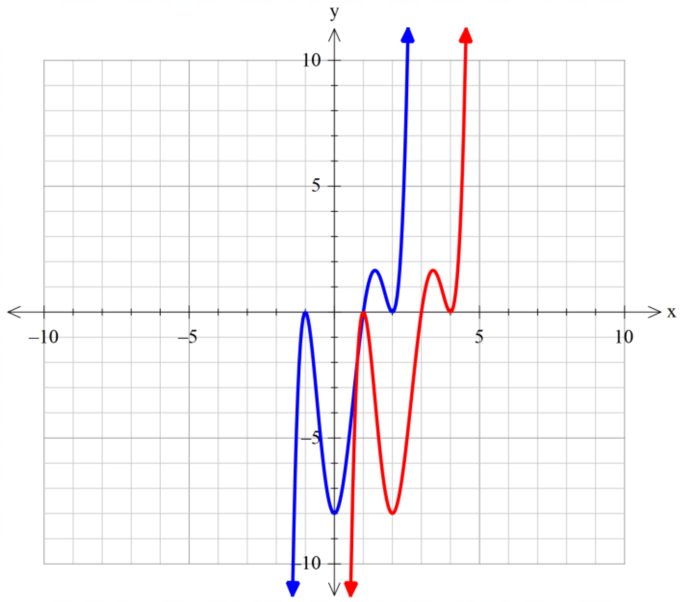
Question 7 **(7 marks)**

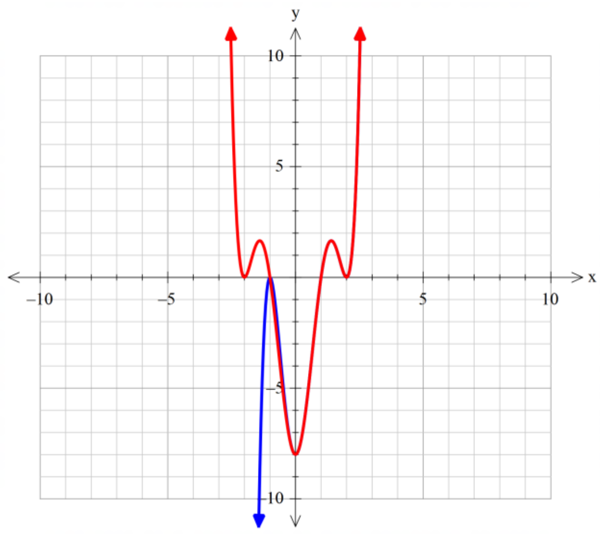
Solution
First note that $$-64\sqrt{2}(1+i) = -128\,cis\left(\frac{i\pi}{4}\right) = 2^7\,cis\left(\frac{5i\pi}{4}\right)$$ Then $z^7 = 2^7 cis\left(i\pi\left[2k+\frac{5}{4}\right]\right) \Rightarrow z = 2cis\left(\frac{i\pi}{7}\left[2k+\frac{5}{4}\right]\right)$ for $k = 0...6$ by de Moivre's theorem Hence the seven roots are $z = 2cis(i\vartheta)$ where $\vartheta = \frac{5\pi}{28}, \frac{13\pi}{28}, \frac{3\pi}{4}, \frac{29\pi}{28}, \frac{37\pi}{28}, \frac{45\pi}{28}, \frac{53\pi}{28}$. Restricting the argument to the stated domain leaves $\vartheta = -\frac{27\pi}{28}, -\frac{19\pi}{28}, -\frac{11\pi}{28}, -\frac{3\pi}{28}, \frac{5\pi}{28}, \frac{13\pi}{28}$ and $\frac{3\pi}{4}$

Mathematical behaviours	Marks
• writes $-64\sqrt{2}(1+i)$ in polar form (1 for modulus, 1 for argument)	1+1
• uses de Moivre's theorem appropriately	1
• writes down the seven required roots (-1 for one mistake)	2
• calculates all the arguments so that they lie in the appropriate given range (-1 for one mistake)	2

Question 8 **(5 marks)**

Question 8(a),(b) **(4 marks)**

<table>
<tr><th colspan="2">Solution</th></tr>
<tr><td colspan="2">(a)


(b)


The graph of $f(x-2)$ is obtained by shifting the graph of $f(x)$ two units to the right.
The graph of $f(|x|)$ is the same as that for $f(x)$ where x is positive, and reflected about the $y-$axis for x negative.</td></tr>
<tr><th>Mathematical behaviours</th><th>Marks</th></tr>
<tr><td>• displays the correct geometric transformations
• plots the graphs reasonably accurately</td><td>1+1
1+1</td></tr>
</table>

Question 8(c) **(4 marks)**

Solution
Commencing with the roots we see that as $f(x)=0$ when $x=-1,1$ and 2 so that $$f(x)=a(x+1)^p(x-1)^q(x-2)^r$$ As the function touches the horizontal axis at $x=-1,2$ then $$f(x)=a(x+1)^2(x-1)(x-2)^2$$ To fix the value of a we note that the curve passes through (0,-8) so that $$a(1)(-1)(4)=-8 \Rightarrow a=2$$ Therefore $f(x)=2(x+1)^2(x-1)(x-2)^2$ Therefore, the equation of the function is $f(x)=2(x+1)^2(x-1)(x-2)^2$

Mathematical behaviours	Marks
• correctly determines the three factors $x+1, x-1$ and $x-2$	1
• identifies the respective powers on the three factors	1
• uses the point (0,-8) to deduce the value of a	1
• correctly determines $a=2$	1